BERNOULLI FLOWS OVER MAPS OF THE INTERVAL

BY

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ABSTRACT

We consider special flows T^t built over shift automorphism φ with a Holder function. We introduce two properties for φ (R and WM) s.t. weak-mixing for T^t implies K if φ is R and it implies Bernoulliness if φ is WM. We apply this to maps of the interval to show that for the Lorentz Attractor Flow with a natural measure weak-mixing implies Bernoulliness.

We turn to the Rohlin–Sinai theory of continuous partitions (see [14], [15], [7]). Let (X,μ) be a probability space. For partitions ξ,η we denote by $\xi\wedge\eta$ the largest partition s.t. if $x\in A\in\xi\wedge\eta$ then $\xi(x)$, $\eta(x)\subset A$ for a.e. $x\in X$. ξ is called discrete if $\mu(C)>0$ for some $C\in\xi$. For $C\subset\xi(x)$ we denote $C\times\eta=\bigcup_{y\in C}\eta(y)$.

DEFINITION 1. (1) A pair $\{\xi,\eta\}$ is called discrete ($\mathcal{D}$) if $\xi\wedge\eta$ is discrete.

(2) An ordered pair $\{\xi,\eta\}$ is called random (R) at $x\in X$ if for each $C\subset\xi(x)$, $\mu(C\mid\xi(x))=1$, we have $\mu(C\times\eta)>0$.

(3) An ordered pair $\{\xi,\eta\}$ is called random if there is $R\subset X$, $\mu(R)>0$ s.t. $\{\xi,\eta\}$ is random at every $x\in R$. R is called a random set for $\{\xi,\eta\}$.

(4) $\{\xi,\eta\}$ is strongly random (SR) if for a.e. $x\in X$ and every $C\subset\xi(x)$, $\mu(C\mid\xi(x))>0$ we have $\mu(C\times\eta)>0$. (Compare with [16].)

Clearly, $\{\xi,\eta\}$ is R implies $\{\xi,\eta\}$ is $\mathcal{D}$.

Anosov flows with non-integrable pair of foliations [1], [2] provides us with a pair $\{\eta^s,\eta^u\}$ s.t. $\eta^s\wedge\eta^u=\{X,\varnothing\}\ \mathrm{mod}_\mu\,0$ and $\{\eta^s,\eta^u\}$ is not random at any $x\in X$.

Let φ be the shift automorphism in a space $X\subset\{1,\cdots,r\}^{\mathbf{Z}}$ of two-sided sequences $x=\{x_i\}_{i=-\infty}^{\infty}$, $(\varphi x)_i=x_{i-1}$, $\varphi(X)=X$, preserving μ. Let $A_r=$

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$\{x \in X: x_0 = r\}$, $\alpha = \{A_1, \cdots, A_r\}$ and $\alpha_m^n = \bigvee_{k=m}^n \varphi^k \alpha$. Assume without loss of generality that $\mu(A) > 0$ for all $A \in \alpha_m^n$, m, $n = 0, 1, \cdots$.

DEFINITION 2. α is called discrete (random, strongly random) if the pair $\{\alpha_{-\infty}^0, \alpha_0^\infty\}$ is discrete (R, SR). An m.p.t. is $\mathcal{D}$(R, SR) if it has a $\mathcal{D}$(R, SR) generator.

The following theorem obviously follows from [15] (see also [7] and our discussion below).

THEOREM 1. *Let α be discrete. Then either φ is a K-automorphism or φ^m is not ergodic for some $m \neq 0$.*

Surely, if α is Bernoulli it is SR. Here is an example of a very weak Bernoulli (VWB) β which is not random at any point (even not discrete).

Let φ be Bernoulli with two-sets generator $\alpha = \{P_1, P_2\}$, $f: X \to R$, $f \mid P_1 = a$, $f \mid P_2 = b$ and a, b be s.t. the special flow T^t built with (φ, f) is Bernoulli. Let $B_1 = \bigcup_{0 \leqq t \leqq a} T^t P_1$, $B_2 = \bigcup_{0 \leqq t \leqq b} T^t P_2$, $\beta = \{B_1, B_2\}$. β is a VWB generator for $T = T^1$. It is easy to see that

$$\beta_{-\infty}^0 = \{T^t x_{-\infty}^0 : 0 \leqq t \leqq a, x \in P_1\} \cup \{T^t x_{-\infty}^0 : 0 \leqq t \leqq b, x \in P_2\},$$

$$\beta_0^\infty = \{T^t x_0^\infty : 0 \leqq t \leqq a, x \in P_1\} \cup \{T^t x_0^\infty : 0 \leqq t \leqq b, x \in P_2\}.$$

Every atom of $\beta_{-\infty}^0 \wedge \beta_0^\infty$ is either $T^t P_1$ for some $0 \leqq t \leqq a$ or $T^t P_2$ for some $0 \leqq t \leqq b$. These atoms all have measure 0 in the space of the flow.

QUESTION. Are there K-automorphisms which are not random (discrete)?

We consider the natural metric ρ in X:

$$\rho(x, y) = \sum_{i \in \mathbf{Z}} 2^{-|i|} e(x_i, y_i),$$

$$e(x_i, y_i) = 0 \quad \text{if} \quad x_i = y_i, \qquad e(x_i, y_i) = 1 \quad \text{if} \quad x_i \neq y_i,$$

and call $G \in \mathcal{F}_\beta$ on $X, G: X \to R$ if G is μ-integrable and $1/G$ is Holder continuous of order $\beta > 0$ (we allow G to grow to ∞).[†]

[†] Actually our proofs below also work for G with

$$\sum_{n=0}^\infty n \cdot \sup_{x, y: x_i = y_i, |i| \leqq n} |G(x) - G(y)| < \infty.$$

We say that α is k-random ($k \geqq 0$) if the pair $\{\alpha_{-\infty}^{0}, \alpha_{-k}^{\infty}\}$ is random. Clearly, m-R implies n-R if $m \geqq n$.

We prove the following theorem.

THEOREM 2. *Let T^t be the special flow built over $(X, \mu, \varphi, \alpha)$ with $0 < G_0 \leqq G \in \mathcal{F}_\beta$. There is $k = k(G)$ depending only on G s.t. if* (1) *G is bounded and α is k-random or* (2) *G is any and α is strongly k-random, then either T^t is a K-flow or T^s is not ergodic for some $s \neq 0$.*

So the statement of the theorem holds for all bounded G if α is k-R for all $k \geqq 0$ and it holds for all G if α is strongly k-R for all $k \geqq 0$.

REMARK. $k(G)$ in the theorem can be estimated from our proofs below. It depends on G_0, β and the Holder coefficient of G.

Gurevič [7] proved such a theorem when α was discrete and G was constant on atoms of $\alpha_{-\infty}^{0} \wedge \alpha_{0}^{\infty}$. We use some of his ideas.

The following property has been responsible for Bernoulliness of Anosov systems (see [4], [6], [13]).

DEFINITION 3. α (or φ) is weak Markov (WM) if given $\varepsilon > 0$ there are $N = N(\varepsilon)$, a set $P = P(\varepsilon)$ of atoms of $\alpha_{-\infty}^{0}$, $\mu(P) > 1 - \varepsilon$ and a set $Q = Q(\varepsilon)$ of atoms of α_{-N}^{∞}, $\mu(Q) > 1 - \varepsilon$ s.t. if $\bar{x}, \bar{y} \in P \cap x_{-N}^{0}$, $x_{-N}^{0} \in \alpha_{-N}^{0}$ then for any set $A \subset Q$ of atoms of α_{-N}^{∞} we have $\mu(A \mid \bar{x}) > 0$ iff $\mu(A \mid \bar{y}) > 0$ and

$$\left| \frac{\mu(A \mid \bar{x})}{\mu(A \mid \bar{y})} - 1 \right| < \varepsilon.$$

We prove

THEOREM 3. *Let α be* WM. *Then α is strongly k-random for all $k \geqq 0$.*

One can easily see that if φ is a K-automorphism and α is WM then α is a weak Bernoulli partition (see [13]). We get from Theorems 3 and 1, and [13]

THEOREM 4. *Let α be* WM. *Then either φ is Bernoulli or φ^m is not ergodic for some $m \neq 0$.*

Using Theorems 2, 3 we apply the Ornstein–Weiss theory [11], [12] to get (see [13])

THEOREM 5. *Let T^t be the special flow over $(X, \alpha, \varphi, \mu)$ with $0 < G_0 \leqq G \in \mathcal{F}_\beta$. Let α be* WM. *Then either T^t is Bernoulli or T^s is not ergodic for some $s \neq 0$.*

In the last section of the paper (which can be read independently), we apply our theorems to maps of the interval and semiflows over such maps.

Suppose that $f: [0,1] \to [0,1]$ is a piecewise C^1-function, i.e. there is a finite partition $0 = a_0 < a_1 < \cdots < a_r = 1$ so that f is differentiable on each (a_i, a_{i+1}). If $\lambda = \inf_{0 \leq x \leq 1} |f'(x)| > 1$ and f is piecewise C^2, i.e. each $f|(a_i, a_{i+1})$ extends to a C^2-function on $[a_i, a_{i+1}]$, then Lasota and Yorke [9] showed that f possesses a smooth invariant measure μ and the density $d\mu/dx$ of such a μ is of bounded variation on $[0,1]$. Using their method, S. Wong [18] has obtained the same result for the case when $\varphi(x) = 1/|f'(x)||(a_i, a_{i+1})$ extends to a function of bounded variation on $[a_i, a_{i+1}]$ (in this case we say that φ is of bounded variation (b.v.) on $[0,1]$). As opposed to the C^2-extension the last condition allows $|f'|$ to grow to ∞. This is especially important since such maps f arise as Poincaré maps for semiflows whose inverse image represents by Williams [17] the Lorenz Attractor Flows ([5], [10]). For the C^2-extension case Rufus Bowen [3] has studied the ergodic properties of (f, μ). He proved that the natural extension of (f, μ) is Bernoulli whenever (f, μ) is weak-mixing and gave some natural conditions for (f, μ) being weak-mixing.

We show that Rufus' results still hold for the bounded variation case. We just reprove his basic lemma 3 and then prove

THEOREM 6. *Let $f: [0,1] \to [0,1]$ be a piecewise C^1-function, $1/|f'|$ be of* b.v., *$\lambda = \inf_{0 \leq x \leq 1} |f'(x)| > 1$ and μ be a smooth f-invariant probability measure. Then the natural extension of f is* WM.

So we get from Theorem 4 and 6

THEOREM 7. *Either the natural extension of (f, μ) is Bernoulli or f^m is not ergodic for some $m \neq 0$.*

Having Rufus' lemma 3 reproved, we repeat his arguments [3] to get

THEOREM 7'. *Let f, μ be as in Theorem 6. Then the natural extension of (f, μ) is Bernoulli if one of the following holds:* (a) $\sup_{n>0} \mu(f^n U) = 1$ *for all nonempty open intervals U with $\mu(U) > 0$;* (b) $r = 2$ *and* $\lambda > \sqrt{2}$; (c) $\lambda > 2$ *and condition* (a) *holds for the sets* $U = (a_j, a_{j+1})$, $1 \leqq j \leqq r-2$.

The Poincaré map (f, μ) in the Williams' construction [17] satisfies (b). So we get

COROLLARY. *The Poincaré map f has only one smooth invariant measure μ (the Wong one* [18]) *and the natural extension of (f, μ) is Bernoulli* (see also Lanford [22] for discussions about f).

We get from Theorems 3 and 5

THEOREM 8. *Let f, μ be as in Theorem 6, $0 < F_0 \leqq F \in \mathcal{F}_\beta$ and S^t be the semiflow built over (f, μ) with F. Then, either the natural extension of S^t is Bernoulli or S^p is not ergodic for some $p \neq 0$.*

A continuous flow T^t on a metric space X is an inverse image of a semiflow S^t on W if there is a continuous surjective map $\psi: X \to W$ s.t. (1) if we denote by $\mathcal{B}$ and $\tilde{\mathcal{B}}$ the Borel σ-algebras in W and X respectively then the smallest T^t-invariant σ-algebra in X which contains $\psi^{-1}(\mathcal{B})$ is all of $\tilde{\mathcal{B}}$ and (2) $\psi T^t = S^t \psi$. If S^t preserves a measure ν on $\mathcal{B}$ then T^t preserves the measure $\tilde{\nu}$ on $\tilde{\mathcal{B}}$, $\tilde{\nu}(\psi^{-1}B) = \nu(B)$, $B \in \mathcal{B}$. This means that T^t in $(X, \tilde{\mathcal{B}}, \tilde{\nu})$ is the natural extension of S^t in $(W, \mathcal{B}, \nu)$.

The Lorenz Attractor Flows (LAF) are by Williams [17] inverse images of semiflows satisfying the conditions of Theorem 8. Providing LAF with the measure $\tilde{\nu}$, we get

THEOREM 9. *The Lorenz Attractor Flow L^t with the invariant measure $\tilde{\nu}$ is either Bernoulli or L^m is not ergodic for some $m \neq 0$.*

Proving Theorem 2 we show that the belonging $G \in \mathcal{F}_\beta$ provides T^t with a pair of partitions $\{\eta^s, \eta^u\}$ analogous to stable and unstable foliations of Anosov flows. We show that if the second alternative of Theorem 2 holds, i.e. T^m is not ergodic for some $m \neq 0$, then the pair has a special property similar to integrability (see [1, 2]) of foliations for Anosov flows. For the LAF the property can be stated as follows.

Let W^s, W^{ss}, W^{uu} be the stable, strong stable and strong unstable foliations of LAF. Let $W^{ss}_\varepsilon(x)$ be the ε-ball centered at x in a leave of W^{ss}. Let $y \in W^{uu}_\delta(x)$. If $\delta, \varepsilon > 0$ are sufficiently small the following map is defined, $p: W^{ss}_\varepsilon(x) \to W^s_{2\varepsilon}(y)$

$$p(z) = W^s_{2\varepsilon}(y) \cap W^{uu}_{2\delta}(z), \qquad z \in W^{ss}_\varepsilon(x).$$

We say that W^{ss} and W^{uu} are integrable if for any $x, \varepsilon, \delta > 0$ small we have $p(W^{ss}_\varepsilon(x)) \subset W^{ss}(y)$.

Apparently, one can prove that, as for Anosov flows, the following alternatives hold for LAF L^t: either $\{W^{ss}, W^{uu}\}$ are non-integrable or L^t is a special flow built with a constant function (suspension). In the first case we have Bernoulliness in Theorem 9 and in the second L^m is not ergodic for some $m \neq 0$. In this last case the spectrum of L^t is not continuous.

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1. Randomess and K-flows

First let us show how the Rohlin–Sinai theory works. Denote $\xi = \alpha_{-\infty}^{0}$ and $\eta = \alpha_{0}^{\infty}$. We have $\varphi^{-1}\xi \leqq \xi$, $\eta \leqq \varphi^{-1}\eta$ and $\bigvee_{n \in Z} \varphi^{n}\xi = \bigvee_{n \in Z}\varphi^{n}\eta = \varepsilon$ where ε is the partition into points. Denote $\xi^{-} = \bigwedge_{n=0}^{\infty}\varphi^{-n}\xi$ and $\eta^{+} = \bigwedge_{n=0}^{\infty}\varphi^{n}\eta$. It follows from [14], [15] that (1) φ is a K-automorphism iff η^{+} (or ξ^{-}) is the trivial partition $N = \{X, \emptyset\}$ $\mathrm{mod}_{\mu} 0$, (2) since the entropy $h(\varphi) < \infty$, $\eta^{+} = \xi^{-} = \pi(\varphi)$ where $\pi(\varphi)$ is the Pinsker partition invariant under φ.

PROOF OF THEOREM 1. We have $\xi^{-} \leqq \xi$, $\eta^{+} \leqq \eta$, $\pi(\varphi) = \xi^{-} \wedge \eta^{+} \leqq \xi \wedge \eta$. Since $\xi \wedge \eta$ is discrete it has a set of positive measure, therefore $\pi(\varphi)$ does. Since $\pi(\varphi)$ is φ-invariant either $\pi(\varphi) = N$ or some degree of φ is not ergodic. □

We are going to prove Theorem 2. Let $G \in \mathcal{F}_{\delta}$ on X, i.e. G is μ-integrable and if $x, y \in x_{-n}^{n} \in \alpha_{-n}^{n}$ then

$$\left| \frac{1}{G(x)} - \frac{1}{G(y)} \right| \leqq H\lambda^{-\delta n} \qquad \text{for some } H > 0 \text{ and } \lambda > 1.$$

Denote $\tilde{K}_n = \{x \in X \colon G(x) > \lambda^{\frac{1}{4}\delta n}\}$, $K_n = X - \tilde{K}_n$. Since G is μ-integrable $\mu(\tilde{K}_n) < C\lambda^{-\frac{1}{4}\delta n}$, $C > 0$. So we get

$$|G(x) - G(y)| \leqq H\lambda^{-\frac{1}{2}\delta n} \qquad \text{for any } x, y \in x_{-n}^{n} \cap K_n.$$

From now on we write $G \in \mathcal{F}_{\beta}$ ($\beta > 0$) if for every $n = 0, 1, 2, \cdots$ there is $K_n \subset X$, $\mu(K_n) > 1 - L\lambda^{-\beta n}$ ($\lambda > 1$) and $|G(x) - G(y)| < L\lambda^{-\beta n}$ whenever $x, y \in x_{-n}^{n} \cap K_n$, $L > 0$.

Let φ be an m.p.t. in (X, μ), $f, g \in L_{\mu}^{1}(X)$ and $\int_X f\, d\mu = \int_X g\, d\mu$. We shall say that f is homologous to g ($f \sim g$) relative φ if there is a measurable function u on (X, μ) s.t. a.e. $f(x) = g(x) + u(x) - u(\varphi^{-1}x)$. (See [8], [13].) Gurevič [8] proved the following lemma.

LEMMA 1. *If $f \sim g$, then the special flows (φ, f) and (φ, g) constructed over (X, μ) with f and g are isomorphic.*

We prove a generalized version of a lemma in [13].

LEMMA 2. *Let φ be the shift automorphism on $X \subset \{1, \cdots, r\}^{Z}$ preserving μ. Let $0 < G_0 \leqq G \in \mathcal{F}_{\beta}$ on (X, μ). Then there is $g \colon X \to R^{+}$ s.t. (1) $g \geqq g_0 > 0$, (2) $g \in \mathcal{F}_{\gamma}$, $\gamma > 0$, (3) $g \sim G$, (4) g is constant on the elements of α_{0}^{∞}.*

(We need (2) for Bernoulliness in paragraph 2 below.)

PROOF. Let $H_n = \bigcap_{i=n}^{\infty} K_i$ (see the definition of $G \in \mathcal{F}_{\beta}$). Then $H_n \subset H_{n+1} \subset$

$\cdots$, $n = 1,2,\cdots$. $\mu(H_n) > 1 - L_1\lambda^{-\beta_1 n}$, $\beta_1 > 0$ and $X = \bigcup_{n=1} H_n \pmod{\mu}$. Let $H_n^1 = \{x \in X : \mu(x_{-n}^n \cap H_n) > 0\}$. Then $\mu(H_n^1) > 1 - L_1\lambda^{-\beta_1 n}$ and for $\tilde{H}_n = \bigcap_{m=n}^{\infty} H_m^1$ we have $\tilde{H}_n \subset \tilde{H}_{n+1} \subset \cdots$, $\mu(\tilde{H}_n) > 1 - L_2\lambda^{-n\delta}$, $\delta > 0$ and $X = \bigcup_{n=1}^{\infty} \tilde{H}_n \pmod{\mu}$. So for a.e. $x \in X$ there is $n(x)$ s.t. $x \in \tilde{H}_n$, $n \geqq n(x)$ and $x \notin \tilde{H}_n$, $n < n(x)$, $n = 1,2,\cdots$. Let $\Delta_n(x) = x_{-n}^n \cap H_n$. Define

$$G_n(x) = G_0 \qquad \text{if } n < n(x),$$

$$G_n(x) = \int_{\Delta_n(x)} G(x)\mu_{\Delta_n}(x) \qquad \text{if } n \geqq n(x),$$

where the integration is performed with respect to the conditional measure $\mu \mid \Delta_n$. The functions G_n are constant on the atoms of α_{-n}^n and $G_n(x) \geqq G_0 > 0$. Set $h_n(x) = G_n(x) - G_{n-1}(x)$. Then $h_k(x) = G_0$, $k < n(x)$, $h_{n(x)}(x) = G_{n(x)}(x) - G_0 \geqq 0$ and $|h_n(x)| \leqq 2L\lambda^{-\beta(n-1)}$. Let m be s.t. $2L\sum_{n=m}^{\infty}\lambda^{-\beta(n-1)} < G_0/2$. We have

$$G(x) = G_m(x) + \sum_{i=m+1}^{\infty} h_i(x). \tag{1}$$

The series in (1) converges a.e. Each h_i is constant on the atoms of α_{-i}^i and the function $\varphi^i h_i(x) = h_i(\varphi^{-1}x)$ is constant on the atoms of α_0^{2i}. Consider the series

$$g^1(x) = G_m(x) + \sum_{i=m+1}^{\infty} \varphi^i h_i(x). \tag{2}$$

Let $\mathscr{D}_n = \bigcap_{i=u}^{\infty} \varphi^i \tilde{H}_i$. Then $\mathscr{D}_n \subset \mathscr{D}_{n+1} \subset \cdots$, $\mu(\mathscr{D}_n) > 1 - L_3\lambda^{-\delta_1 n}$, $\delta_1 > 0$ and $X = \bigcup_{n=1}^{\infty} \mathscr{D}_n \pmod{\mu}$. So for a.e. $x \in X$ there is $d = d(x) > 0$ s.t. $x \in \mathscr{D}_n$, $n \geqq d$ or $\varphi^{-i}x \in \tilde{H}_i$, $i \geqq d$ and $|\varphi^i h_i(x)| \leqq 2L\lambda^{-\beta(n-1)}$, $i > d$. It follows that the series in (2) converges a.e. and by our choice of m, $g'(x) \geqq G_0/2 = g_0 > 0$. g' is constant on the atoms of α_{-m}^{∞}. Show that $g' \in \mathscr{F}_\gamma$ for some $\gamma > 0$. Let $y, z \in x_{-n}^n \cap \mathscr{D}_{n/2}$, $n \geqq m$. Since $\varphi^i h_i$ is constant on the atoms of α_0^{2i}, $\varphi^i h_i(y) = \varphi^i h_i(z)$ for $i \leqq n/2$ and $|\varphi^i h_i(y)|$, $|\varphi^i h_i(z)| \leqq 2L\lambda^{-\beta(i-1)}$, $i > n/2$. We have

$$|g'(y) - g'(z)| = \left| \sum_{i=[n/2]+1}^{\infty} (\varphi^i h_i(y) - \varphi^i h_i(z)) \right| \leqq 4L \sum_{i=[n/2]}^{\infty} \lambda^{-\beta(i-1)}.$$

This says that $g' \in \mathscr{F}_\gamma$, $\gamma > 0$.

Consider the series

$$u(x) = \sum_{i=m+1}^{\infty} \sum_{k=0}^{i-1} \varphi^k h_i(x). \tag{3}$$

Let $E_i = \bigcap_{k=0}^{i-1} \varphi^k \tilde{H}_i$, $\mu(E_i) > 1 - L_2 i\lambda^{-\delta i}$. Let $B_n = \bigcap_{j=n}^{\infty} E_j$, $B_n \subset B_{n+1} \subset \cdots$,

$\mu(B_n) > 1 - L_2 \Sigma_{i=n}^{\infty} i\lambda^{-i\delta}$ and $X = \bigcup_{n=1}^{\infty} B_n \pmod{\mu}$. So for a.e. $x \in X$ there is $b = b(x)$ s.t. $x \in E_j$, $j \geqq b$ or $|\varphi^k h_j(x)| \leqq 2L_2\lambda^{-\delta(j-1)}$, $k = 0, \cdots, j-1$, $j \geqq b$. So the series in (3) converges a.e. It is readily seen that $G(x) = g'(x) + u(x) - \varphi u(x)$ a.e. and so $G \sim g'$. Then the function $g = \varphi^m g'$ is in $\mathcal{F}_\gamma$, constant on atoms of α_0^∞, $g \geqq g_0 > 0$ and $g \sim g' \sim G$. □

REMARKS. (1) In the same way we may construct such g constant on atoms of $\alpha_{-\infty}^0$.

(2) It is easy to see from the proof that the function $u \in \mathcal{F}_{\gamma'}$, for some $\gamma' > 0$.

Lemma 2 says that we may assume G constant on atoms of $\alpha_{-\infty}^0$. The flow T^t acts in the space $W = \{(x, y): x \in X,\ 0 \leqq y \leqq G(x),\ (x, G(x)) = (\varphi x, 0)\}$ and preserves the measure ν: $d\nu = d\mu \times dt/\bar{G}$, where $\bar{G} = \int_X G d\mu$. The fact $G \in \mathcal{F}_\beta$ provides W with a pair of partitions analogous to stable and unstable foliations for Anosov flows. To see this let us explore the geometric meaning of the function u in the proof of Lemma 2. $G(x) = g(x) + u(x) - \varphi u(x)$ and g is constant on α_0^∞. We assume for simplicity that the sets K_n in the definition of $G \in \mathcal{F}_\beta$ are sets of atoms of α_{-n}^n. It follows from the construction of u that there are sets E_n of atoms of α_{-n}^∞, $n = 0, 1, \cdots$, s.t. $\mu(E_n) > 1 - M\lambda^{-\gamma n}$, $|u(\varphi^{-k}x) - u(\varphi^{-k}y)| < M\lambda^{-\gamma k}$ for all $k \geqq n$, $x, y \in x_{-n}^\infty \in E_n$, $M > 0$ and $\varphi^{-k}E_n \subset E_{n+k}$, $k \geqq 0$. Let $n(x) = \min\{n: x \in x_{-n}^\infty \in E_n\}$ and let ξ be the partition of X into sets $\xi(x) = x_{-n(x)}^\infty$. We have $\varphi^{-1}\xi \geqq \xi$. Let $\xi_n = \xi \vee \alpha_{-n}^\infty$. Denote $V(x) = \{(z, u(z)), z \in \xi_k(x)\}$. If k is sufficiently large then for every $x \in X$ there is $s = s(x) > 0$ s.t.

$$V_s = T^s V(x) \subset \{(z, y): z \in \xi_k(x)\}.$$

For $w = (x, y) \in W$ define

$$\eta^u(w) = \begin{cases} T^{-u(\varphi x) - G(x) + y} V(\varphi x) & G(x) \geqq y > u(x) + s(x), \\ \\ T^y V(x) & 0 \leqq y \leqq u(x) + s(x). \end{cases}$$

Since $G(x) - u(\varphi x) + u(x)$ is constant on atoms of ξ_k the partition η^u is well-defined and $T^t\eta^u \geqq \eta^u$, $t \leqq 0$ and $\vee_t T^t\eta^u = \varepsilon$. η^u is analogous to unstable foliations of Anosov flows. One can see that if $y, z \in \eta^u(w)$ then $d(T^t y, T^t z) \to 0$, $t \to -\infty$ where d is the natural metric in $W \subset X \times R$.

Let us define a stable partition η^s. G is constant on $\alpha_{-\infty}^0$. For $w = (x, y) \in W$, $0 \leqq y < G(x)$ define $\eta^s(w) = T^y x_{-\infty}^0$. Then $T^t\eta^s \geqq \eta^s$, $t \geqq 0$, $\vee_t T^t\eta^s = \varepsilon$ and $d(T^t y, T^t z) \to 0$, $t \to \infty$, $y, z \in \eta^s(w)$.

REMARK. We have already made the first step in choosing the $k(G)$ in Theorem 2. It should be so large that we could have the set V_s as above.

LEMMA 3. *Let* (1) G *be bounded and* $\{\alpha^0_{-\infty}, \alpha^{\infty}_{-k}\}$ *be* R *or* (2) G *be anything and* $\{\alpha^0_{-\infty}, \alpha^{\infty}_{-k}\}$ *be* SR. *Then if* k *is sufficiently large* $\{\alpha^0_{-\infty}, \xi_k\}$ *is random.*

PROOF. (1) If G is bounded there is $k_0 > 0$ s.t. $\xi(x) \in \alpha^{\infty}_{-l}$ for some $l \leqq k_0$, $x \in X$. Therefore if $k \geqq k_0$, a random set for $\{\alpha^0_{-\infty}, \xi_k\}$ is that of $\{\alpha^0_{-\infty}, \alpha^{\infty}_{-k_0}\}$. (2) We have $\xi_k = \xi \vee \alpha^{\infty}_{-k}$. Let $A = \{x \in X : \xi_k(x) \in \alpha^{\infty}_{-k}\}$. By the construction of ξ, $\mu(A) > 1 - M\lambda^{-\gamma k}$. Let k be s.t. $1 - M\lambda^{-\gamma k} > 0$ and $R = \{x \in A : \mu(A \mid x^0_{-\infty}) > 0\}$, $\mu(R) > 0$. Let $C = A \cap x^0_{-\infty}$, $x \in R$. By SR, $\mu(C \times \xi_k) = \mu(C \times \alpha^{\infty}_{-k}) > 0$. So R is a random set for $\{\alpha^0_{-\infty}, \xi_k\}$. □

REMARK. We made our second choice of $k(G)$ in the lemma.

PROOF OF THEOREM 2. Denote $\eta^{u}_{+} = \wedge_{t \geq 0} T^t \eta^{u}$, $\eta^{s}_{-} = \wedge_{t \leq 0} T^t \eta^{s}$. $\eta^{u}_{+} = \eta^{s}_{-} = \pi(T^t) = \pi$ where π is the Pinsker partition invariant under T^t (see [7], [15], [16]), $\pi \leqq \eta^{s} \wedge \eta^{u}$.

We should prove that either $\pi = N$ (trivial) or T^b is not ergodic for some $b \neq 0$.

Working with continuous partitions we should be careful with sets of measure 0, i.e. our arguments should not depend upon any changes by such sets.

Denote $I_1 = \{w \in W :$ there is $C \in \eta^{s} \wedge \eta^{u}$ s.t. $w \in C$ and $\eta^{u}(w), \eta^{s}(w) \subset C\}$, $\nu(I_1) = 1$.

$\pi \leqq \eta^{s} \wedge \eta^{u}$ means that there is $I_2 \subset W$, $\nu(I_2) = 1$ s.t. if $w \in I_2$ then there are $C(w) \in \eta^{s} \wedge \eta^{u}$ and $C_\pi(w) \in \pi$ s.t. $w \in C(w)$ and $C(w) \subset C_\pi(w)$.

π is invariant under T^t means that there is $I_3 \subset W$, $\nu(I_3) = 1$ s.t. if $w \in I_3$ then $w \in C_\pi$ for some $C_\pi \in \pi$, $T^t w \in C'_\pi$ for some $C'_\pi \in \pi$, $t \in R$ and $C'_\pi = T^t C_\pi$ for all $t \in R$.

Let $I_4 = I_1 \cap I_2 \cap I_3$, $\nu(I_4) = 1$, $I = \{w \in I_4 : \nu(I_1 \mid \eta^{s}(w)) = 1\}$, $\nu(I) = 1$ and $\tilde{\eta}^{s}(w) = \eta^{s}(w) \cap I_1$.

If $w \in I$ then (1) $\eta^{s}(w)$, $\eta^{u}(w) \subset C(w) \subset C_\pi(w)$ for some $C = C(w) \in \eta^{s} \wedge \eta^{u}$, $C_\pi = C_\pi(w) \in \pi$, (2) $T^t w \in C'_\pi = T^t C_\pi \in \pi$, $t \in R$, (3) for any $v \in \tilde{\eta}^{s}(w)$, $\eta^{u}(v) \subset C(w)$ and therefore $\tilde{\eta}^{s}(w) \times \eta^{u} \subset C(w) \subset C_\pi(w)$, (4) $\tilde{\eta}^{s}(w) \times \eta^{u} \subset I_3$.

We are given that the pair $\{\alpha^0_{-\infty}, \alpha^{\infty}_{-k}\}$, $k = K(G)$ is R or SR in (1) or (2) of the Theorem. By Lemma 3 the pair $\{\alpha^0_{-\infty}, \xi_k\}$ is R (R also denotes a random set for the pair, $\mu(R) > 0$).

Denote $\theta = \bigcup_{t=p}^{q} T^t R$ where $0 < p < q$ are such that $\theta \subset \{(u, v) \in W : u \in R\}$. Divide θ into sets $\{l(z) = \bigcup_{t=p}^{q} T^t z,\ z \in R\}$. Since $\nu(I) = 1$ and $\nu(\theta) > 0$ there is

$l = l(\bar{z})$, $\bar{z} \in R$ s.t. the Lebesgue measure $\lambda(I/l) = 1$. Let $Q = I \cap l$ and $B = \bigcup_{w \in Q} \bar{\eta}^s(w) \times \eta^u$. $B \subset I_3$ by (4) above. The projection of $\bar{\eta}^s(w) \times \eta^u$, $w \in Q$ on X has a form $A \times \xi_k$ for some $A = A(w) \subset \bar{z}^0_{-\infty}$, $\mu(A/\bar{z}^0_{-\infty}) = 1$. Since $\bar{z} \in R$, $\mu(A \times \xi_k) > 0$ and therefore $\nu(B) > 0$.

Suppose that T^t is ergodic and $\pi > N$. Then there is a set D of atoms of π s.t. $0 < \nu(D) < 1$. (It is clear when we consider D as a set of atoms of π and when as a subset of W.) Since $\nu(B) > 0$ we may take the D s.t. $\nu(B \cap D) > 0$. Let $\bar{D} = \{C_\pi \in D : \nu(B/C_\pi) > 0)$, $0 < \nu(\bar{D}) \leqq \nu(D) < 1$.

Let C'_π, $C''_\pi \in \bar{D}$ and $x' \in C'_\pi \cap B$, $x'' \in C''_\pi \cap B$. We have $x' \in \bar{\eta}^s(w') \times \eta^u \subset C'_\pi$, $x'' \in \bar{\eta}^s(w'') \times \eta^u \subset C''_\pi$, $w', w'' \in Q$ and $w'' = T^a w'$ for some $a : |a| \leqq q - p$. Since $w', w'' \in I$, $C''_\pi = T^a C'_\pi$. Such a number a exists for any two atoms of $\bar{D}$.

Since T^r is ergodic and $\nu(B \cap \bar{D}) > 0$ there is $r > q - p > 0$ s.t. $\nu(T^r(B \cap \bar{D}) \cap (B \cap \bar{D})) > 0$. Let $x \in T^r(B \cap \bar{D}) \cap (B \cap \bar{D})$, $x \in C'_\pi \in \bar{D}$ and $T^{-r}x \in C''_\pi \in \bar{D}$. Since $x \in B \subset I_3$ we have $C'_\pi = T^r C''_\pi$. Let $|a| \leqq q - p$ be s.t. $C''_\pi = T^a C'_\pi$ and $b = a + r > 0$. Then $T^b C'_\pi = C'_\pi$. Since such an a exists for any two atoms of $\bar{D}$ we get $T^b C_\pi = C_\pi$ for every $C_\pi \in \bar{D}$. So $T^b \bar{D} = \bar{D}$. Since $0 < \nu(\bar{D}) < 1$ this implies that T^b is not ergodic. □

Let

$$w' \in \eta^s(w) \times \eta^u, \qquad w' = (x, y),$$

$$Q(w') = \{(u, v) \in W : u \in x^0_{-\infty}\}, \qquad \zeta(w, w') = Q(w') \cap (\eta^s(w) \times \eta^u).$$

Definition. The pair $\{\eta^s, \eta^u\}$ is called integrable if $\zeta(w, w') \subset \eta^s(w')$ for a.e. $w, w' \in W$.

Proposition. *If T^b is not ergodic for some $b \neq 0$, then $\{\eta^s, \eta^u\}$ is integrable.*

Proof. If T^b is not ergodic for some $b \neq 0$ then there is an eigenfunction

$$f(T^t w) = e^{i\lambda t} f(w) \qquad \text{a.e.} \tag{4}$$

for all t, $\lambda \neq 0$. One can see from the proof of Theorem 2 that f is constant on atoms of η^s and of η^u. Therefore f is constant on $\eta^s(w) \times \eta^u$ and on $\zeta(w, w')$. But (4) shows then that $\zeta(w, w') \subset \eta^s(w') \bmod_\nu 0$. □

2. WM and Bernoulliness

Proof of Theorem 3. We have to prove that for a.e. $x \in X$, all $p \geqq 0$, all $F \subset x^0_{-\infty}$, $\mu(F \mid x^0_{-\infty}) > 0$ we have $\mu(F \times \alpha^\infty_{-p}) > 0$.

Let $\varepsilon_n = 2^{-n}$ and P_n, Q_n, N_n be as in the definition of WM. So $\mu(P_n), \mu(Q_n) > 1 - 2^{-n}$ and for $\bar{x}, \bar{y} \in \alpha^0_{-N_n} \cap P_n$, $A \subset Q_n$, $\mu(A \mid \bar{x}) > 0$ iff $\mu(A \mid \bar{y}) > 0$, $\bar{x} = x^0_{-\infty}$.

Denote
$$A_k = \{x \in X : \mu(Q_k \mid \bar{x}) > 1 - 2^{-k/2}\},$$
$$B_k = \{x \in X : \mu(x^0_{-N_k} \cap P_k) > 0\},$$
$$C_k = \{x \in X : \bar{x} \in P_k\}.$$

We have $\mu(A_k) > 1 - 2^{-k/2}$, $\mu(B_k), \mu(C_k) > 1 - 2^{-k}$. Let
$$\tilde{A}_n = \bigcap_{k=n}^{\infty} A_k, \qquad \tilde{B}_n = \bigcap_{k=n}^{\infty} B_k, \qquad \tilde{C}_n = \bigcap_{k=n}^{\infty} C_k.$$
Then $\mu(\tilde{A}_n) > 1 - 2^{-n/4}$, $\mu(\tilde{B}_n)$, $\mu(\tilde{C}_n) > 1 - 2^{-(n-2)}$. Let
$$A = \bigcup_{n \geqq 0} \tilde{A}_n, \qquad B = \bigcup_{n \geqq 0} \tilde{B}_n, \qquad C = \bigcup_{n \geqq 0} \tilde{C}_n,$$
$\mu(A \cap B \cap C) = 1$ and if $x \in A \cap B \cap C$ then there is $q(x) > 0$ s.t. $x \in A_k \cap B_k \cap C_k$ for all $k \geqq q(x)$. So if $k \geqq q(x)$ then
$$\mu(x^0_{-N_k} \cap P_k) > 0, \quad \bar{x} \in P_k \quad \text{and} \quad \mu(Q_k \mid \bar{x}) > 1 - 2^{-k/2}.$$

Let $p \geqq 0$ be fixed, $x \in A \cap B \cap C$ and $F \subset \bar{x}$, $\mu(F \mid \bar{x}) > 0$. Let $k \geqq q(x)$ be s.t. $N_k \geqq p$ and $\mu(F \cap Q_k \mid \bar{x}) > 0$. Denote $E = F \cap Q_k$ and $E' = E \times \alpha^{\infty}_{-N_k}$. Let $\mathcal{D} = P_k \cap x^0_{-N_k}$, $\mu(\mathcal{D}) > 0$. If $\bar{y} \in \mathcal{D}$ then by WM $\mu(E' \mid \bar{y}) > 0$ and we have
$$\mu(E') = \int_{\bar{y} \subset x^0_{-N_k}} \mu(E' \mid \bar{y}) d\mu \geqq \int_{\bar{y} \in \mathcal{D}} \mu(E' \mid \bar{y}) d\mu > 0. \qquad \square$$

Proof of Theorem 5. In view of Theorems 2 and 3 it is enough to show that if T' is a K-flow then T' is Bernoulli. To get it we just modify slightly the proof of theorem 3.1 in [13].

Instead of T' built over $(X, \alpha, \mu, \varphi)$ with $G \in \mathcal{F}_\beta$ we consider the flow $\tilde{T}'$ built with $g \in \mathcal{F}_\gamma$ from Lemma 2. By Lemma 1, T' and $\tilde{T}'$ are isomorphic. Let $t_0 > 0$ and $T = \tilde{T}^{t_0}$. We show that T is Bernoulli. Let K_n be as in the definition of $g \in \mathcal{F}_\gamma$ and $L_n = \bigcap_{i=n}^{\infty} \varphi^{-i} K_i$, $\tilde{L}_n = X - L_n$, $\mu(\tilde{L}_n) < \lambda^{-\gamma_1 n}$, $\gamma_1 > 0$ for big n. Since g is μ-integrable given $\varepsilon > 0$ there is $\delta > 0$ s.t. $\int_A g d\mu < \varepsilon$ whenever $\mu(A) < \delta$. If now n is s.t. $\lambda^{-\gamma_1 n} < \delta$ then we have $\nu(V_n) < \varepsilon$ where $V_n = \{(x, y) \in W : x \in \tilde{L}_n\}$. This enables us to repeat arguments in [13] by noting that lemma 3.3 in [13] is now true outside an additional set of small ν-measure. $\square$

3. Maps of the interval and semiflows

Henceforth f is a piecewise C^1-map of $[0, 1] = I$, $\lambda = \inf |f'(x)| > 1$, $\varphi(x) = 1 / |f'(x)|$ is of bounded variation on I and μ is a smooth f-invariant measure on I. By Wong's theorem the density $p(x) = d\mu(x)/dx$ is of bounded variation on I.

$\mathcal{P}$ will always denote the partition $\mathcal{P} = \{(0, a), \cdots, (a_{r-1}, 1)\}$ into intervals of continuity of f. One can see that $\bigvee_{n=0}^{k} f^{-n}\mathcal{P}$ is a partition into intervals of length $\leqq \lambda^{-k}$.

The following lemma is proved in Bowen's paper [3].

LEMMA 3.1. *If* $A \in \bigvee_{n=0}^{k} f^{-n}\mathcal{P}$, $A \neq \emptyset$, *and* $\bar{A} \cap \{a_0, \cdots, a_r\} = \emptyset$ *then* $fA \in \bigvee_{n=0}^{k-1} f^{-n}\mathcal{P}$.

LEMMA 3.2. *There is a constant* $L > 0$ *s.t. given* $\varepsilon > 0$ *if* $N > L \log(1/\varepsilon)$ *then there is a collection of atoms* $\alpha_N \subset \bigvee_{n=0}^{N} f^{-n}\mathcal{P}$ *so that* $\mu(\cup \alpha_N) > 1 - \varepsilon$ *and for any* $x, y \in A \in \alpha_N$ *we have*

$$\frac{p(x)}{p(y)} \in [e^{-\varepsilon}, e^{\varepsilon}] \qquad \textit{and} \qquad \frac{\varphi(x)}{\varphi(y)} \in [e^{-\varepsilon}, e^{\varepsilon}].$$

For the function p the lemma is proved in Bowen's paper. Following his way we'll demonstrate the proof for φ.

PROOF. Since f maps I into itself $\int_I |f'(x)| dx < \infty$ and since the density p is bounded $C_1 = \int_I |f'(x)| d\mu(x) < \infty$. This implies that for any $l > 0$

$$\mu\left\{x \in I: |f'(x)| > \frac{1}{l}\right\} = \mu\{x \in I: \varphi(x) < l\} < C_1 l. \tag{3.1}$$

Consider the following exhaustive list of possibilities for an atom $A \in \bigvee_{n=0}^{N} f^{-n}\mathcal{P}$ and $\delta > 0$.

(1) $\varphi(x) \geqq \delta/2$ for all $x \in A$ and $\varphi(y) > e^{\delta}\varphi(z)$ for some $y, z \in A$.

(2) $\varphi(x) \leqq \delta/2$ and $\varphi(y) \geqq 3\delta/4$ for some $x, y \in A$.

(3) $\varphi(x) \leqq 3\delta/4$ for all $x \in A$.

(4) $\varphi(x) \geqq \delta/2$ for all $x \in A$ and $\varphi(y) \leqq e^{\delta}\varphi(z)$ for all $y, z \in A$.

Let K be the total variation of $\varphi(x)$ on I. The variation of $\varphi(x)$ over an A satisfying (1) or (2) is at least $\gamma = \min\{(e^{\delta} - 1)\delta/2, \delta/4\} > \delta^2/4$. The total number of such atoms A is at most $K\gamma^{-1}$ and the total μ-measure of such atoms is at most $K \cdot \gamma^{-1}\lambda^{-N}\|p\|_\infty < 4K\|p\|_\infty\delta^{-2}\lambda^{-N}$. The total μ-measure of all atoms satisfying (3) is at most $C_1\frac{3}{4}\delta$ by (3.1). So if we denote by α_N the collection of all the atoms satisfying (4) then the total μ-measure of the atoms which $\notin \alpha_N$ is at most $C_2(\delta^{-2}\lambda^{-N} + 3\delta/4)$ where $C_2 = \max\{4K\|p\|_\infty, C_1\}$. If $\lambda^{-N} < \delta^3/8$ or $N > L \log(1/\delta)$ for some $L > 0$ then the last μ-measure is at most $C_2\delta$. We complete the proof taking $\delta = \min\{\varepsilon, \varepsilon/C_2\}$. □

Picking $\varepsilon = 2^{-\sqrt{N}}$ in Lemma 3.2 we get the following

COROLLARY 1. *There is* $N_0 > 0$ *s.t. if* $N > N_0$ *then there is a collection of atoms* $\alpha_N \subset \bigvee_{n=0}^{N} f^{-n}\mathcal{P}$ *s.t.* $\mu(\cup \alpha_N) > 1 - 2^{-\sqrt{N}}$ *and for any* $x, y \in A \in \alpha_N$ *we have*

$$(3.2)\qquad \left|\frac{p(x)}{p(y)}-1\right|<4^{-\sqrt{N}} \quad \textit{and} \quad \left|\frac{\varphi(x)}{\varphi(y)}-1\right|<4^{-\sqrt{N}}.$$

We denote $\tilde{\alpha}_N=\{A\in\bigvee_{n=0}^{N}f^{-n}\mathcal{P}: A\not\in\alpha_N\}$. So $\mu(\cup\tilde{\alpha}_N)<2^{-\sqrt{N}}$.

LEMMA 3.3 (basic). *Given $\varepsilon>0$, there is an $M=M(\varepsilon)$ so that for each $m\geqq 0$ one can find a collection of atoms $\beta=\beta_{m+M}\subset\bigvee_0^{m+M}f^{-n}\mathcal{P}$ with*

(1) $f^mB\in\bigvee_{n=0}^{M}f^{-n}\mathcal{P}$ *for* $B\in\beta$,

(2) $\mu(\cup\beta)>1-\varepsilon$,

(3) $\left|\dfrac{\mu(f^m\tilde{B})}{\mu(f^mB)}-\dfrac{\mu(\tilde{B})}{\mu(B)}\right|<\varepsilon\dfrac{\mu(\tilde{B})}{\mu(B)}$

for any measurable $\tilde{B}\subset B\in\beta$, $\mu(B)>0$.

PROOF. Using Lemma 3.1, R. Bowen showed that the set $\tilde{\beta}$ of those $B\in\bigvee_{n=0}^{m+M}f^{-n}\mathcal{P}$ for which (1) does not hold has total μ-measure at most $\mathcal{D}\lambda^{-M}$ for some $\mathcal{D}>0$. Take $M>N_0$ and the collections $\tilde{\alpha}_N$ (see Corollary 1) for $N=M, M+1,\cdots,M+m$. Denote

$$\tilde{\beta}_k=f^{-m+k}\tilde{\alpha}_{M+k},\qquad k=0,1,\cdots,m.$$

Each atom of $\tilde{\beta}_k$ is composed with some atoms of $\bigvee_{n=0}^{M+m}f^{-n}\mathcal{P}$. Since μ is f-invariant we get from Corollary 1 that the total μ-measure of atoms in all the $\tilde{\beta}_k$, $k=0,1,\cdots,m$ is at most $\Sigma_{k=M}^{\infty}2^{-\sqrt{k}}$. Denoting $\beta=\{B\in\bigvee_{n=0}^{M+m}f^{-n}\mathcal{P}: B\not\in\tilde{\beta}$ and $B\not\in\tilde{\beta}_k$, $k=0,1,\cdots,m\}$ and picking $M>N_0$ so large that $\max\{\mathcal{D}\lambda^{-M},\Sigma_{k=M}^{\infty}2^{-\sqrt{k}}\}<\varepsilon/2$ we get $\mu(\cup\beta)>1-\varepsilon$ and (1) holds for the β. So if $B\in\beta$ then $f^mB\in\bigvee_{n=0}^{M}f^{-n}\mathcal{P}$ and $f^m|_B$ is one-to-one. In addition, if $B\in\beta$, then $B\not\in\tilde{\beta}_k$, $k=0,1,\cdots,m$ and therefore $f^kB\subset A\in\alpha_{M+m-k}$. Then by (3.2)

$$(3.3)\qquad \left|\frac{\varphi(f^kx)}{\varphi(f^ky)}-1\right|<4^{-\sqrt{M+m-k}},\qquad \left|\frac{p(f^kx)}{p(f^ky)}-1\right|<4^{-\sqrt{M+m-k}}$$

for all $k=0,1,\cdots,m$ and any $x,y\in B\in\beta$. We have for $\tilde{B}\subset B\in\beta$

$$\mu(f^m\tilde{B})=\int_{f^m\tilde{B}}p(y)dy=\int_{\tilde{B}}p(f^mx)|(f^m)'(x)|dx$$

$$=\int_{\tilde{B}}\frac{p(f^mx)|(f^m)'(x)|}{p(x)}p(x)dx=\int_{\tilde{B}}q(x)p(x)dx$$

where

$$\frac{q(x)}{q(y)}=\frac{p(f^mx)}{p(f^my)}\cdot\frac{p(y)}{p(x)}\cdot\frac{|(f^m)'(x)|}{|(f^m)'(y)|}=\frac{p(f^mx)}{p(f^my)}\cdot\frac{p(y)}{p(x)}\prod_{k=1}^{m}\frac{\varphi(f^ky)}{\varphi(f^kx)}.$$

Since the product $\prod_{n=1}^{\infty}(1 \pm 4^{-\sqrt{n}})$ converges and therefore

$$\lim_{M\to\infty}\prod_{k=0}^{\infty}(1 \pm 4^{-\sqrt{M+k}}) = 1$$

we get from (3.3) that if $B \in \beta$ and $x, y \in B$ then

$$\left|\frac{q(x)}{q(y)} - 1\right| < \varepsilon \qquad \text{or} \qquad |q(x) - q(y)| < \varepsilon q(y)$$

for sufficiently large M. Fixing $y \in B \in \beta$ we get

$$\frac{\mu(f^m\tilde{B})}{\mu(f^mB)} = \frac{\int_{\tilde{B}} q(x)p(x)dx}{\int_B q(x)p(x)dx}$$

$$< \frac{q(y)\int_{\tilde{B}} p(x)dx + \varepsilon q(y)\int_{\tilde{B}} p(x)dx}{q(y)\int_B p(x)dx - \varepsilon q(y)\int_B p(x)dx}$$

$$= \frac{\mu(\tilde{B})}{\mu(B)}\frac{1+\varepsilon}{1-\varepsilon}.$$

Similarly

$$\frac{\mu(f^m\tilde{B})}{\mu(f^mB)} > \frac{\mu(\tilde{B})}{\mu(B)}\frac{1-\varepsilon}{1+\varepsilon}.$$

Obviously, these imply condition (3) of the lemma. □

REMARKS. From now we denote $\mathcal{P}_M = \bigvee_{n=0}^{M} f^{-n}\mathcal{P}$.

(1) One can see from the proof of Lemma 3.3 that if $B \in \beta_{M+m}$ then $f^kB \in \beta_{M+m-k}$ for all $0 \leqq k \leqq m$, $m \geqq 0$.

(2) Assertion (3) of the lemma can be rewritten as

$$\left|\frac{\mu(\tilde{B})}{\mu(f^m\tilde{B})} - \frac{\mu(V)}{\mu(f^mB)}\right| < \varepsilon\frac{\mu(\tilde{B})}{\mu(f^m\tilde{B})}. \tag{3.4}$$

(3) Let $B \in \beta_{M+m}$, $A \subset B$ and $A \in \mathcal{P}_{M+m+k}$, $m, k \geqq 0$. It follows from (3.4) that

$$\left|\frac{\mu(A)}{\mu(f^{-m}f^mA)} - \frac{\mu(B)}{\mu(f^{-m}f^mB)}\right| < \varepsilon\frac{\mu(A)}{\mu(f^{-m}f^mA)}$$

since μ is f-invariant.

(4) Let $\mathbf{C}_m$ be the collection of atoms $C \in \mathcal{P}_M$ so that at least $1-\sqrt{\varepsilon}$ (in terms of μ-measure) of the atoms $B \in \mathcal{P}_{M+m}$ with $f^m B \subset C$ satisfy $B \in \beta_{M+m}$. Then $\mu(\cup \mathbf{C}_m) \geqq 1-\sqrt{\varepsilon}$. Let $\mathbf{C}(\varepsilon)=\{C \in \mathcal{P}_M: C \in \mathbf{C}_m$ for infinitely many $m \geqq 0\}$. Since $\mu(\cup \mathbf{C}_m) \geqq 1-\sqrt{\varepsilon}$ for all m we have $\mu(\cup \mathbf{C}(\varepsilon)) \geqq 1-\sqrt{\varepsilon}$. It follows from Remark (1) that actually if $C \in \mathbf{C}_m$ then $C \in \mathbf{C}_k$ for all $0 \leqq k \leqq m$ and therefore $\mathbf{C}(\varepsilon)=\{C \in \mathcal{P}(M)$: $C \in \mathbf{C}_m$ for all $m \geqq 0\}$.

Taking $M=M(\varepsilon^2)$ we get

(5) Given $\varepsilon>0$ there are $M>0$ and a set $\mathbf{C}(\varepsilon)$ of atoms $C \in \mathcal{P}_M \mu(\cup \mathbf{C}(\varepsilon)) \geqq 1-\varepsilon$ with the following property: for any $m \geqq 0$ there is a set $\beta_{M+m} \subset \mathcal{P}_{M+m}$, $\mu(\cup \beta_{M+m}) \geqq 1-\varepsilon$ s.t. if $B \in \beta_{M+m}$ and $\mu(B \cap f^{-m}C)>0$ for some $C \in \mathbf{C}(\varepsilon)$ then for any $A \subset C$, $A \in \mathcal{P}(M+k), k \geqq 0$, $\mu(B \cap f^{-m}A)>0$ and

$$\left|\frac{\mu(B \cap f^{-m}A)}{\mu(f^{-m}A)}-\frac{\mu(B)}{\mu(f^{-m}B)}\right|<\varepsilon \frac{\mu(B \cap f^{-m}A)}{\mu(f^{-m}A)} .$$

(6) It is easy to see from the proof of Lemma 3.3 that $M(\varepsilon)$ can be taken as $\mathcal{D} \log(1/\varepsilon)$ for some $\mathcal{D}>0$.

We now look at the natural extension of (f, μ).

Denote $A=\{0, a_0, a_1, \cdots, 1\}$, $\Delta=\bigcup_{k,n \geqq 0} f^{-k} f^n A$ and $\mathcal{G}=I-\Delta$. Then $\mu(\mathcal{G})=1$, $f(\mathcal{G})=\mathcal{G}$ and if $z \in \mathcal{G}$ then $z=\bigcap_{n=0}^{\infty} f^{-n} P_n$, $P_n \in \mathcal{P}$, $n=0,1, \cdots$, $\mathcal{P}=\{A_1, \cdots, A_r\}$ for a unique sequence $\{\cdots P_2 P_1 P_0\}=z_{-\infty}^0 \in\{A_1, \cdots, A_r\}^{\mathbf{Z}}$. Also $(fz)_{-\infty}^0=\{\cdots P_2 P_1\}$.

Define Ω by $\Omega=\{\omega \in\{1, \cdots, r\}^{\mathbf{Z}^-}: \exists z_\omega=z \in \mathcal{G}$ s.t. $z_{-\infty}^0=\{\cdots P_{\omega_{-1}} P_{\omega_0}\}$, $\psi: \Omega \rightarrow \mathcal{G}$ by $\psi(\omega)=z_\omega$ and $\tilde{f}: \Omega \rightarrow \Omega$ by $(\tilde{f}\omega)_i=\omega_{i-1}$. ψ is a one-to-one measurable map and $f\psi=\psi\tilde{f}$. Let $\tilde{\mathcal{P}}=\psi^{-1}\mathcal{P}$, then $\tilde{\mathcal{P}}_m^n=\bigvee_{k=m}^n \tilde{f}^{-k}\tilde{\mathcal{P}}=\psi^{-1}\mathcal{P}_m^n$. For $A \in \tilde{\mathcal{P}}_m^n$ define $\tilde{\mu}(A)=\mu(\psi A)$ and extend $\tilde{\mu}$ to an $\tilde{f}$-invariant measure on the σ-algebra $\tilde{\mathcal{B}}$ in Ω generated by the cylindric sets $A \in \tilde{\mathcal{P}}_m^n$, $0 \leqq m \leqq n<\infty$. Then ψ is an isomorphism between (f, μ) in I and $(\tilde{f}, \tilde{\mu})$ in Ω. So the natural extension of (f, μ) is that of $(\Omega, \tilde{\mathcal{B}}, \tilde{\mu}, \tilde{f})$ which we denote by $(X, \mathcal{B}, \bar{\mu}, \varphi)$. This means that $X \subset\{1, \cdots, r\}^{\mathbf{Z}}$ is a set of two-sided sequences $x=\{x_i\}_{i=-\infty}^{\infty}$, $\varphi: X \rightarrow X$ is the shift automorphism $(\varphi x)_i=x_{i-1}$, $\varphi(X)=X$, $\bar{\mu}$ is a φ-invariant measure on the σ-algebra $\mathcal{B}$ in X, generated by cylindric sets and the map $\pi: x \rightarrow x_{-\infty}^0$ is a measurable map from X onto Ω s.t. $\mathcal{B}$ is generated by $\pi^{-1}(\tilde{\mathcal{B}})$ and $\bar{\mu}(\pi^{-1}B)=\tilde{\mu}(B)$, $B \in \tilde{\mathcal{B}}$. The partition $\alpha=\pi^{-1}\tilde{\mathcal{P}}$ is a generator for φ, let $\alpha_m^n=\bigvee_{k=m}^n \varphi^k \alpha$.

Proof of Theorem 6. Remark (5) above says that $(X, \bar{\mu}, \varphi)$ has the following property:

Given $\varepsilon>0$ there is $M=M(\varepsilon)>0$ and a set P of atoms $x_{-M}^0 \in \alpha_{-M}^0$, $\bar{\mu}(P) \geqq 1-\varepsilon$ s.t. for any $m \geqq 0$ there is a set Q_m of atoms $x_{-M}^m \in \alpha_{-M}^m$,

$\bar{\mu}(Q_m) \geqq 1-\varepsilon$ s.t. if $x_{-M}^{m} \in Q_m$ and $x_{-M}^{m} \in x_{-M}^{0} \subset P$ then for every $x_{-k}^{0} \subset x_{-M}^{0}$, $k \geqq M$ we have

$$\left| \frac{\bar{\mu}(x_{-M}^{m}/x_{-k}^{0})}{\bar{\mu}(x_{-M}^{m}/x_{-M}^{0})} - 1 \right| < \varepsilon. \tag{3.5}$$

Since this is true for every $x_{-k}^{0} \subset x_{-M}^{0} \in P$, $k \geqq M$ it follows that if $x_{-M}^{m} \in Q_m$, $B \in P$ and $\bar{x}, \bar{\bar{x}} \in \alpha_{-\infty}^{0} \cap B$ then $\bar{\mu}(x_{-M}^{m} | \bar{x}) > 0$ iff $\bar{\mu}(x_{-M}^{m} | \bar{\bar{x}}) > 0$ and

$$\left| \frac{\mu(x_{-M}^{m}/\bar{x})}{\mu(x_{-M}^{m}/\bar{\bar{x}})} - 1 \right| < \varepsilon.$$

Now let $Q = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} Q_m$. Then $\bar{\mu}(Q) = \lim_{n\to n} \bar{\mu}(\bigcup_{m=n}^{\infty} Q_m) \geqq 1-\varepsilon$ and Q is a set of atoms of α_{-M}^{∞}. Let $A \subset Q$, $\bar{\mu}(A) > 0$ be a set of atoms of α_{-M}^{∞}. Then for all n, $A \subset \bigcup_{m=n}^{\infty} Q_m$ and therefore there is a sequence $i_1 < i_2 < \cdots$ s.t. $A \subset Q_{i_j}$, $j = 1, 2, \cdots$ or sets $A_{i_j} \subset Q_{i_j}$ of atoms of $\alpha_{-M}^{i_j}$ s.t. $A_{i_1} \supset A_{i_2} \supset \cdots$ and $A = \bigcap_{j=1}^{\infty} A_{i_j}$. Since $\bar{\mu}(A | \bar{x}) = \lim_{j\to\infty} \bar{\mu}(A_{i_j}/\bar{x})$ we get from (3.5) that if $B \in P$, $\bar{x}, \bar{\bar{x}} \in \alpha_{-\infty}^{0} \cap B$ then $\bar{\mu}(A | \bar{x}) > 0$ iff $\bar{\mu}(A | \bar{\bar{x}}) > 0$ and

$$\left| \frac{\bar{\mu}(A | \bar{x})}{\bar{\mu}(A | \bar{\bar{x}})} - 1 \right| < \varepsilon.$$

Finally we summarize that $(X, \bar{\mu}, \varphi)$ has the following property:

Given $\varepsilon > 0$ there are $M = M(\varepsilon) > 0$, a set P of atoms of $\alpha_{-\infty}^{0}$, $\bar{\mu}(P) \geqq 1-\varepsilon$ and a set Q of atoms of α_{-M}^{∞}, $\bar{\mu}(Q) \geqq 1-\varepsilon$ s.t. for all $x_{-M}^{0} \in \alpha_{-M}^{0}$, all $\bar{x}, \bar{\bar{x}} \in P \cap x_{-M}^{0}$ and any set $A \subset Q$ of atoms of α_{-M}^{∞} we have $\mu(A | \bar{x}) > 0$ iff $\mu(A | \bar{\bar{x}}) > 0$ and

$$\left| \frac{\mu(A | \bar{x})}{\mu(A | \bar{\bar{x}})} - 1 \right| < \varepsilon.$$

This is exactly the WM property. □

Proof of Theorem 8. We have $X \xrightarrow{\pi} \Omega \xrightarrow{\psi} I$ where the maps π, ψ are defined above. Define $G: X \to R^+$ by $G(x) = F(\psi\pi x)$. It is clear that $G \in \mathscr{F}_\gamma$ on X for some $\gamma > 0$. T' over $(X, \bar{\mu}, \varphi)$ built with G is the natural extension of S'. So we apply Theorems 3 and 5. □

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